Estimates of the total gravitation radiation in the head-on black hole collision

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Abstract

We report on calculations of the total gravitational energy radiated in the head-on black hole collision, where we use the geometry of the Robinson-Trautman metrics.

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1 Introduction

Accurate description of the dynamics of relativistic gravitating systems remains one of the challenging problems in general relativity. There is great interest in particular on the description of systems involving very compact objects, as for example black holes. It is expected that these kind of systems will be describable in terms of an appropriate isolated model; which in a relativistic theory of gravity is expressed by an asymptotically flat spacetime. In this description the gravitational radiated energy will be encoded in the behavior of the physical field at future null infinity.

In numerical calculations these ideas have been used in attacking the problem of the time symmetric asymptotically flat initial data corresponding to a two equal mass black hole system[1][2]. It would be nice to be able to compare these calculations with some analytic solution of the field equation; but in trying to do this one is faced with the fact that there is up to now only one known vacuum asymptotically flat radiating family of solutions, the so called Robinson-Trautman metrics[3]. These spacetimes represent the dynamics of a single black hole which settles down, in the asymptotic future[4], to the Schwarzschild geometry.

In the process of two black holes falling into each other in a head-on collision, one can distinguish two natural eras: one in which the two black holes are separated and falling, and the other in which they already form a single black hole which is decaying to a stationary state. Since the Robinson-Trautman metrics represent a black hole, one could estimate the total gravitational radiation emitted after the formation of the black hole using these geometries. In doing this one needs to associate the data of the two black hole collision system to the Robinson-Trautman data. We do this by matching, at an arbitrary retarded time, the radiation content of these metrics with the radiation calculated from the quadruple formula with Newtonian dynamics at the moment of collapse.

2 Power of the energy radiated from Newtonian dynamics + quadrupole formula

Let us consider the Newtonian problem of two particles of mass m_0, which start at rest at an initial distance d_i. One can estimate the power of the energy carried away by gravitational waves when they are at a distance d assuming Newtonian dynamics and the quadrupole approximation. More specifically; the quadrupole radiation formula relates the power of the energy radiated with the time derivatives of the quadrupole:

$$P_Q = \frac{1}{5} \sum_{i,j=1}^{3} \left(\frac{d^3 Q_{ij}}{dt^3} \right)^2$$

where

$$Q_{ij} = q_{ij} - \frac{1}{3} \delta_{ij} \; q,$$

$$q_{ij} = \int T^{00} x^i x^j d^3 x,$$

and we are using geometric units in which the Newton's gravitational constant and the speed of light have the unit value. Then, the total power radiated is

$$P_Q(d) = \left(\frac{1}{15} \right) \frac{1}{\left(\frac{d}{M_0} \right)^4} \left(\frac{1}{\left(\frac{d}{M_0} \right)} - \frac{1}{\left(\frac{d_i}{M_0} \right)} \right),$$

where $M_0 \equiv 2 \; m_0$ is the total mechanical mass and d the distance at observation.

3 Robinson-Trautman metrics: the quadrupole approximation

In a previous paper[4] it was found the general asymptotic behavior of the Robinson-Trautman metrics in the asymptotic future. The leading behavior of these metrics is governed by the quadrupole structure of the spacetime.

More explicitly, the whole geometry of these metrics is determined by a scalar $V(u,\theta,\phi)$, where u, can be considered a retarded time and (θ,ϕ) the coordinates of a 2-sphere. According to Ref. [4], for an axisymmetric quadrupole structure, the asymptotic behavior of this scalar, for $u \to \infty$ is

$$V = 1 + A\, e^{-\frac{2u}{M_\infty}}\, Y_{20} + O(e^{-\frac{4u}{M_\infty}}),$$

where A is a constant M_∞ is the asymptotic mass of the spacetime in the regime $u \to \infty$, and Y_{20} is a spherical harmonic function. This geometry undoubtedly represents the dynamics of a single black hole; which becomes evident when in the limit for $u \to \infty$, i.e., V=1, one recognizes the Schwarzschild metric.

As we have described above, there are two natural eras in the process of the head-on collision: before and after the formation of the remaining black hole. Since the Robinson-Trautman geometries describe the dynamics of a single black hole, it is natural to try to study, with these geometries, the final stage of the process.

The time derivative of the Bondi shear of the Robinson-Trautman sections of future null infinity has the following asymptotic behavior:

$$\dot{\sigma}_0 = \sqrt{6}\, A\, e^{-\frac{2u}{M_\infty}}\, {}_2Y_{20} + O(>),$$

where now ${}_2Y_{20}$ is a spin weight 2 spherical harmonic function and $O(>)$ means higher order terms of the time exponential. With this information one can calculate the flux of gravitational Bondi radiation at future null infinity in first order, obtaining:

$$P_B = \frac{6}{4\pi}\, A^2\, e^{-\frac{4u}{M_\infty}}.$$

To relate the Robinson-Trautman geometry with the problem of the head-on collision of two black holes, we identify the Bondi power at some retarded time with the power calculated from the quadrupole formula at the moment when the two black holes touch; that is when $d = 2M_0$. In this way we determine A appearing in the above formulae.

With the Bondi flux one can calculate in first order the total energy radiated, namely:

$$\begin{aligned} E = \int_{u_0}^{\infty} P_B(u)\, du &= \frac{6}{4\pi} A^2 \frac{M_\infty}{4} e^{4u/M_\infty}\Big|_{\infty}^{u_0} \\ &= \frac{6}{4\pi} A^2 \frac{M_\infty}{4} e^{-4u_0/M_\infty} = P_B(u_0)\frac{M_\infty}{4}. \end{aligned}$$

Let M_i denote de mass of the spacetime at the *initial* retarded time; then one has the relation

$$M_\infty = M_i - E.$$

Therefore, the ratio radiation over initial mass, can be expressed by

$$\frac{E}{M_i} = \frac{M_\infty \frac{P_B(u_0)}{4}}{M_\infty \left(1 + \frac{P_B(u_0)}{4}\right)} = \frac{\frac{P_B(u_0)}{4}}{\left(1 + \frac{P_B(u_0)}{4}\right)}.$$

Identifying the Bondi with the quadrupole flux one would obtain

$$\frac{E}{M}=\frac{\left(\frac{1}{60}\right)\frac{1}{l_f^4}\left(\frac{1}{l_f}-\frac{1}{l}\right)}{1+\left(\frac{1}{60}\right)\frac{1}{l_f^4}\left(\frac{1}{l_f}-\frac{1}{l}\right)},$$

with $l_f \equiv \frac{d_f}{M}$ and $l \equiv \frac{d_i}{M}$. In this expression we identify M with M_i ; that is we map the Newtonian total mass to the total mass of the spacetime at the initial retarded time.

4 Time symmetric 2-black hole initial data

In reference [5] it was presented the initial data corresponding to a time symmetric 2-black hole system. There the ADM mass M_M was calculated, and a magnitude D_M (there denoted with capital l) was defined, having the interpretation of some measure of initial distance between the black holes. More precisely D_M was defined as the length of the minimal line from the "ring" at minimum radius from one throat to the respective "ring" in the other throat.

If one wants to compare calculations coming from different realizations of the same physical situation, namely the head-on collision of two equal mass black holes, one must provide a relation between the physical quantities involved. The two fundamental physical quantities involved in this situation are the total mass of the system and the separation of the black holes. As we have done in the previous section, the identification of the masses is straight forward, since the natural thing to do is to identify the invariantly defined total masses in each case. There remains however the relation among the notion of distance involved in each case.

The question is: Is there a natural way to identify the Misner magnitude D_M with the Newtonian notion of distance appearing in the estimates of the total gravitational energy radiated using the quadrupole formula approach? A simple identification of D_M with d would be too naive in the regime of small distances. We seek then for some relation between D_M and d such that it captures the most important physical aspects of both systems.

In order to do this let us recall that in reference [6] the time symmetric initial data for a two equal mass black hole system was represented by the 3-geometry:

$$ds^2=\left(1+\frac{m_{1\infty}}{2r_1}+\frac{m_{2\infty}}{2r_2}\right)^4 ds_F^2,$$

where ds_F^2 is the line element of a flat 3-geometry, r_1 and r_2 are the Euclidean distances from the field point to the deleted points in the flat 3-geometry which represent the location of the first and second black hole respectively, and $m_{1\infty}$ and $m_{2\infty}$ are parameters which characterize the spacetime. If in the conformal, scalar appearing in the above line element, one eliminates the third term, the constant time Schwarzschild line element is obtained, with mass $m_{1\infty}$ expressed in terms of isotropic coordinates.

According to the arguments of reference [6] the mass of, let us say, the first black hole is

$$m_1 = m_{1\infty} + \frac{m_{1\infty}\, m_{2\infty}}{2r_{12}},$$

where now r_{12} is the Euclidean distance between the black holes; while the total mass of the spacetime is

$$M = m_1 + m_2 - \frac{m_{1\infty}\, m_{2\infty}}{r_{12}}.$$

This expression agrees with the first relativistic correction to the Newtonian total mass; since in first order one has

$$M_{rel} \equiv M_0 + E_0 = m_0 + m_0 - \frac{m_0^2}{d_i}.$$

Therefore it seems that the Newtonian distance, for large d, should be identified with the coordinate distance of the flat 3-geometry of the time symmetric problem. So using the line element ds^2 shown above for the region between the black holes, and identifying the radius of each black hole with double their respective masses, we define the distance between the black holes by

$$D = 2\, m_1 + \int_{2m_1}^{d-2m_2} \left(1 + \frac{m_{1\infty}}{2r} + \frac{m_{2\infty}}{2(d-r)}\right)^2 dr + 2\, m_2,$$

which in terms of the total mass M for a two equal mass system can be expressed by

$$D = 2\, M\left(1 + \frac{M}{4d}\right) + \int_{M\left(1+\frac{M}{4d}\right)}^{d-M\left(1+\frac{M}{4d}\right)} \left(1 + \frac{m_\infty}{2r} + \frac{m_\infty}{2(d-r)}\right)^2 dr;$$

then, defining $L \equiv \frac{D}{M}$ and $l \equiv \frac{d}{M}$ one has the relation

$$L = 2\,\left(1 + \frac{1}{4l}\right) + \int_{\left(1+\frac{1}{4l}\right)}^{l-\left(1+\frac{1}{4l}\right)} \left(1 + \frac{1}{4\,\lambda} + \frac{1}{4(l-\lambda)}\right)^2 d\lambda.$$

It is observed that with respect of this variable, the black holes do not touch exactly at $l = 2$ but when $l^* = 2\left(1 + \frac{1}{4\, l^*}\right)$; which corresponds to $L = l^* = 2.22$. This picture is in agreement with the numerical calculations since it is found in reference [2] that for $L < 2.49$ a common apparent horizon surrounds both holes [1].

We relate d to the Misner magnitude D_M by setting $D(d) = D_M$.

[1] The threshold mentioned in their reference is $\mu < 1.36$ which corresponds to our $L < 2.49$.

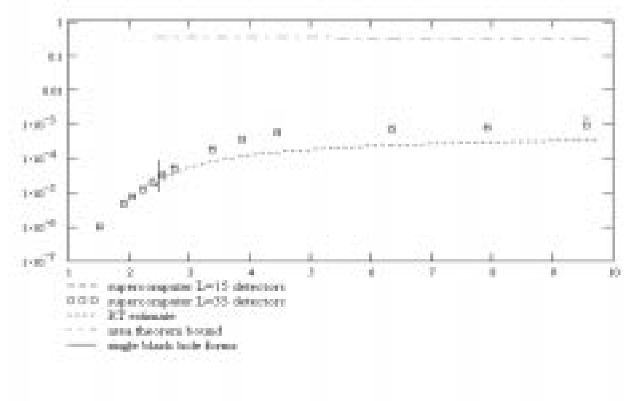

Figure 1: The total gravitational energy radiated away in the process in units of initial total mass. The two sets of numerical data correspond to the location of the detectors at $L = 15$ and $L = 35$ respectively. The vertical bar at approximately $L = 2.49$ corresponds to the region where a single black hole is formed. The RT estimate is plotted in terms of the relation $L(l)$ explained in the main text.

5 Upper limit for the radiation from the area theorem

One can estimate an upper limit for the total energy radiated away in the form of gravitational waves when two very separated black holes collide, from the area theorem; which using the relation between total mass and single black hole mass shown above and in terms of the quantities already defined, establishes that

$$\frac{E_{rad}}{M} \leq \left(1 - \frac{1}{\sqrt{2}}\right)\left(1 + \frac{1}{4l}\right).$$

6 Comparison of estimates with supercomputer results

In ref. [2] the numerical calculation of the total energy radiated for the time symmetric head-on collision of two equal mass black holes was reported. They improved previous calculations [1] with the help of latest generation of supercomputers, and further developed analytic and numerical techniques. In the next graph we plot their recent[7] data, our Robinson-Trautman (RT) estimate for the total energy radiated and the upper limit from the area theorem.

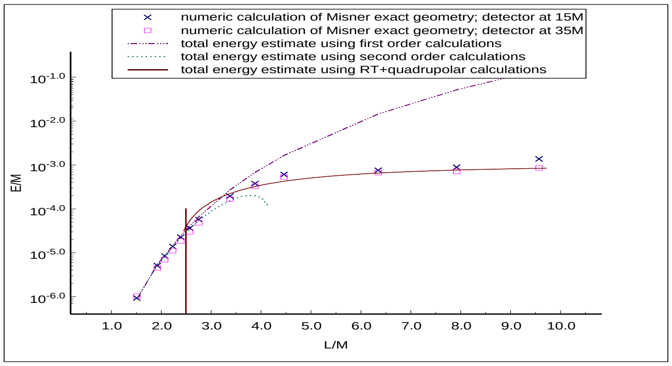

Figure 2: Same as Fig. 1 but now replacing the plain RT estimate by a line that shows the total gravitational energy radiated away in the whole process, which is the sum of the RT energy estimate after the formation of the remaining black hole plus the quadrupole Newtonian estimate of the energy radiated before the formation of the remanent black hole.

It can be observed that the RT estimate keeps bellow the supercomputer data; but this is no surprise since one had expected to only estimate, by this mean, the total energy radiated after the formation of the resulting black hole.

The energy radiated in the first stage, when the black holes fall from d_i to d, before the formation of the resulting black hole, can be estimated from the quadrupole formula and using the Newtonian dynamics; that is

$$E_Q = \int_{t_{d_i}}^{t_d} P_Q\left(x(t)\right) dt = \int_{d_i}^{d} P_Q(x) \frac{1}{\dot{x}}\, dx;$$

from which it is deduced that

$$\frac{E_Q}{M} = \left(\frac{1}{15\sqrt{2}}\right)\left(\frac{2}{7}\left(\frac{1}{l_f}-\frac{1}{l}\right)^{\frac{7}{2}} + \frac{4}{5l}\left(\frac{1}{l_f}-\frac{1}{l}\right)^{\frac{5}{2}} + \frac{2}{3l^2}\left(\frac{1}{l_f}-\frac{1}{l}\right)^{\frac{3}{2}}\right);$$

with $l \equiv \frac{d_i}{M}$, $l_f \equiv \frac{d_f}{M}$ and M is the total mass of the system.

Then adding to the RT estimate the quadrupole Newtonian estimate one obtains the total energy radiated during the entire process in the quadrupole approximation, which is shown in the next graph.

This remarkable agreement between our estimates and the supercomputer calculations allows one to think that the global quantities, as the total energy radiated, for a gravitating system like the one under consideration, is not very dependent on the model realization of the system; since, in particular, the Robinson-Trautman approach is not time symmetric. Therefore the supercomputer results obtained from the time symmetric two black holes head-on collision system might have a wider range of validity than the one assumed up to now.

It is important to note that our estimates can be calculated for values of L corresponding to distance over mass relation above the threshold of the formation of the remaining black hole; which from reference [2] it is deduced to be around L=2.49. So our estimates are complementary of those calculated in reference [8] in which, using a completely different approach, they have estimated with high precision the total energy radiated for small L; while their method overestimates the total energy in two orders of magnitude in the vicinity of $L \approx 8$ (which correspond to their $\mu_0 \approx 3$).

The total gravitational energy radiated in the large separation regime was represented through semianalytic method in reference [2] by their equation (6). The approach followed in that reference requires of a series of extrapolation in the applicability of known equations. First of all they use an equation for the total energy radiated calculated in the test particle limit[9] (eq. (1)); that is, one mass much smaller that the other mass ($\tilde{m} << \tilde{M}$) falling from infinity, and then they extrapolate the equation to the equal mass case ($\tilde{m} = \tilde{M}$). This approach needs from several correcting factors. Using an altered quadrupole radiation formula they suggest a correcting factor, appearing in their eq. (4), that takes into account the effects associated to the fact that the infall is not from infinity but from a finite distance. The alteration in the quadrupole formula comes from a substitution of the radial velocity by a "nonunique" [2] expression. It is observed that the integral in their eq. (4) is calculated up to the upper limit $2\tilde{M}$, which is consistent with the test particle limit approach, but it overestimates the total gravitational energy radiation in the equal mass case. Therefore they have to introduce another correcting factor smaller than one, there called F_h, which is intended to take into account the black hole structure of both objects. These points were also studied in references [10] and [11]. It is important to emphasize that we use, without alterations, the quadrupole radiation formula for the first stage before the collision, up to the point in which the two horizons touch each other ($d \approx 4\tilde{M}$); this in contrast to the case of a test particle crossing one horizon ($d \approx 2\tilde{M}$). Then, in order to take into account the black hole structure of the remaining object we use the Robinson-Trautman geometry, which radiation is to be added to the quadrupole estimate; this in contrast to the reducing factor F_h.

The present estimate of the total gravitational energy radiated is based on a couple of simple ideas with direct physical interpretation; which is a contribution to the understanding of the global aspects of the head-on black hole collision.

It would be interesting to reproduce these calculations taken into account post-Newtonian techniques, as those used in reference [12], to see whether these effects will make any substantial difference to our calculations.

A question that motivated us for this work was how a global quantity, as the total energy radiated, could be dependent on the details of the modeling of a physical situation. On purpose we use a completely different approach, and the result is that this global quantity does not appear to be very dependent on the details of the modeling.

[2] See reference [14] of reference [2].

As a final comment let us point out that since the time symmetric initial data obviously incorporates incoming radiation in the system, it is important to be able to know whether this incoming radiation will have an important contribution to the outgoing radiation. Since the Robinson-Trautman geometry can be considered to represent pure outgoing radiating spacetimes, and also the quadrupole estimate only takes into account retarded fields, the agreement of our estimates with the numerical calculations, of the time symmetric problem, suggests that the incoming radiation contribution to the outgoing one is negligible.

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